BIBLIOGRAPHY

1. Neimark, Iu. I. and Fufaev, N. A., Dynamics of Nonholonomic Systems. M., "Nauka", 1967.
2. Quanjel, J., Les équations générales de la méchanique dans le cas des liaisons non-holonomes. Rend. de Circ. Mat. di Palermo, t. 22, 1906.
3. Ghori, Q. K., Hamilton-Jacobi theorem for nonlinear nonholonomic dynamical systems. ZAMM, Bd. 50, H. 9, 1970.
4. Arzhanykh, I. S., Impulse Field. Tashkent, "Nauka", 1965.
5. Chetaev, N. G., On Poincaré equations. PMM Vol. 5, №2, 1941.
6. Levi-Civita, T. and Amaldi, U., Compendio di Meccanica Razionale. Bologna, 1928.

Translated by L. K.

UDC 531

PERIODIC SOLUTIONS FOR EQUATIONS OF CERTAIN AUTONOMOUS SYSTEMS

PMM Vol. 36, №6, 1972, pp. 1114-1117

E. D. ZHITEL'ZEIF
(Leningrad)

(Received January 29, 1971)

The sufficient conditions are established for the existence of a stable limit cycle for systems of the form

$$x^{\cdot} = y, \quad y^{\cdot} = -g(x) - f_0(x)\,y - \ldots - f_n(x)\,y^n$$

for $n = 2$ and $n = 2m + 1$. The conditions of the theorem of existence and uniqueness of the solution are assumed to hold.

1. Consider the system

$$x^{\cdot} = y, \quad y^{\cdot} = -y\,[f_1(x)\,y + f_0(x)] - g(x) \tag{1.1}$$

introducing the notation

$$F_1(x) = \exp\left(\int_0^x f_1(x)\,dx\right), \quad F(x) = 2\int_0^x F_1(x)\,f_0(x)\,dx - \lambda\int_0^x \frac{dx}{F_1(x)}$$

$$r(x) = 2\int_0^x F_1^2(x)\,g(x)\,dx + \int_0^x F_1(x)\,F(x)\,f_0(x)\,dx$$

$$Q(x) = r(x) - \frac{1}{4}F^2(x), \qquad G(x) = \int_0^x g(x)\,dx$$

Theorem 1. System (1.1) has at least one stable limit cycle, provided that the following conditions hold:

1. Numbers $a < b < 0 < c < d$ and $\lambda > 0$ exist such, that the functions $F(x)$ and $g(x)$ have the following consecutive signatures:

$$\begin{array}{llll} g(x) < 0 & \text{for } x \in (a, 0), & g(x) > 0 & \text{for } x \in (0, d) \\ F(x) < 0 & \text{for } x \in (a, b), & F(x) > 0 & \text{for } x \in (b, 0) \\ F(x) < 0 & \text{for } x \in (0, c), & F(x) > 0 & \text{for } x \in (c, d) \end{array}$$

2. $M = \min \{Q(a), Q(d)\} > Q(x) + [\sqrt{-\lambda^{-1}F_1^3(x)F(x)g(x)} + {}^1/_2|F(x)|]^2$
for $x \in [b, c]$

3. $f_0(0) < 0$

Proof. Let us consider the family of curves

$$\Phi(x, y) = F_1^2(x)y^2 + F_1(x)F(x)y + r(x) = C \quad (1.2)$$

From this we have

$$y_{1,2} = \frac{1}{F_1(x)}\left[-\frac{1}{2}F(x) \pm \sqrt{C - Q(x)}\right] \quad (1.3)$$

Let $m = \sup \{Q(x)\}$ for $b \leqslant x \leqslant c$ and $M = \min \{Q(a), Q(d)\}$. Using arguments identical to those in [1] we can show that for $C \in (m, M]$ Eq. (1.2) defines a family of simple closed curves enclosing each other and the coordinate origin and such, that C increases on passing from the inner to the outer curves. Differentiating (1.2) we obtain, by virtue of system (1.1) and noting that $F_1'(x) - F_1(x)f_1(x) = 0$,

$$d\Phi / dt = -\lambda y^2 - F_1(x)F(x)g(x) \quad (1.4)$$

We shall show that on the curve $\Phi(x, y) = M$, $d\Phi / dt < 0$.

In accordance with Condition 1 we have, on the intervals (a, b) and (c, d), $F_1(x)F(x)g(x) > 0$. Consequently $d\Phi / dt < 0$ on the parts of the curve lying between the straight lines $x = a$ and $x = b$, and between $x = c$ and $x = d$.

Let us determine the sign of $d\Phi / dt$ on the interval $[b, c]$ on the upper arc $y = y_1(x)$, and on the lower arc $y = y_2(x)$ of the curve, with $y_1(x)$ and $y_2(x)$ defined by (1.3) in which the plus and minus signs are taken appropriately.

Let $b < x < 0$. Then $F(x) > 0$ and from the Condition 2 we have

$$y_1(x) > \sqrt{-\lambda^{-1}F_1(x)F(x)g(x)}$$

The latter, together with (1.4), imply that on the arc $y = y_1(x)$, $d\Phi/dt < 0$. Further, since $F(x) > 0$, then $-y_2(x) > y_1(x)$. Consequently, $d\Phi / dt < 0$ on the arc $y = y_2(x)$. Thus $d\Phi / dt < 0$ for $x \in (b, 0)$ on both the upper and lower arc of the curve $\Phi(x, y) = M$. We can show in a similar manner that $d\Phi / dt < 0$ on this curve also in the interval $(0, c)$.

Let us consider another family of curves

$$\varphi(x, y) = {}^1/_2\, y^2 + G(x) = C \quad (1.5)$$

It is clear that this is also a family of simple closed curves enclosing each other and the coordinate origin and such, that C increases on the passage from the inner to the outer curves. Differentiating (1.5) we obtain, by virtue of (1.1),

$$d\varphi / dt = -y^2[yf_1(x) + f_0(x)]$$

At the point $O(0, 0)$ we obtain $yf_1(x) + f_0(x) = f_0(0) < 0$ (according to Condition 3). Consequently $d\varphi / dt \geqslant 0$ on the curves belonging to (1.5) and corresponding to sufficiently small C.

Taking a curve belonging to (1.5) on which $d\varphi / dt \geqslant 0$ and the curve $\Phi(x, y) = M$ on which $d\Phi / dt < 0$, we obtain an annulus contained between these two curves, and all trajectories of the system (1.1) are directed into this annulus. Since the annulus has no singularities, it contains at least one stable limit cycle.

The proof shows clearly that $d\Phi / dt < 0$ not only on the curve $\Phi(x, y) = M$, but also on any curve $\Phi(x, y) = C$, where $C \in (m, M]$, provided that the inequality in Condition 2 holds when M in this condition is replaced by C. In particular, $d\Phi / dt < 0$ on the curve $\Phi(x, y) = N$, where

$$N = \sup \{Q(x) + [\sqrt{-\lambda^{-1} F_1^3(x) F(x) g(x)} + {}^1/_2 |F(x)|]^2\}, \qquad \text{if } N > 0$$

Thus we obtain a certain estimate for the position of the limit cycle on the phase plane. It lies in the region bounded by the curve $\Phi(x, y) = N$.

2. Let us consider the system

$$x^\cdot = y, \qquad y^\cdot = -yf(x, y) - g(x) \tag{2.1}$$

$$f(x, y) = \sum_{k=0}^{2m} f_k(x) y^k$$

Theorem 2. System (2.1) has at least one stable limit cycle, provided that the following conditions hold:

1. A number K exists such, that $f_{2m}(x) > K > 0$ for all x,
2. $\lim\limits_{x \to \pm\infty} x^{-(2m-k)} f_k(x) = 0, \quad k = 1, 2, \ldots, 2m - 1$
3. $f_0(0) < 0, \quad f_0(x) \to +\infty \quad$ for $x \to +\infty$ and $x \to -\infty$
4. $xg(x) > 0 \quad$ for $x \neq 0, \qquad \int\limits_0^x g(x)\,dx \to +\infty \quad$ for $x \to \pm\infty$

Proof. Assume that the topographic family for the system (2.1) is represented by

$$\Phi(x, y) = \frac{1}{2} y^2 + \int\limits_0^x g(x)\,dx = C$$

Differentiating this equation we obtain, by virtue of the system (2.1),

$$d\Phi / dt = -y^2 f(x, y) \tag{2.2}$$

We shall show that a number $R > 0$ exists such, that $f(x, y) > 0$ on any circumference $x = r\cos\theta$, $y = r\sin\theta$ for which $r > R$. At the points of the circumference we have

$$f(x, y) = \sum_{k=0}^{2m} r^k \sin^k\theta f_k(r\cos\theta) = F(r, \theta) \qquad \begin{pmatrix} 0 \leqslant r < +\infty \\ 0 \leqslant \theta \leqslant 2\pi \end{pmatrix} \tag{2.3}$$

Let us inspect the sign of this function in the upper semiplane, i.e. for $0 \leqslant \theta \leqslant \pi$.

By the condition that $f_0(\pm\infty) = +\infty$ we can find a number r_1 such, that for $r > r_1$, the inequalities $F(r, 0) > 0$ and $F(r, \pi) > 0$ hold. Since the function $F(r, \theta)$ is continuous, a number θ_1 exists such, that when $r > r_1$, $F(r, \theta) > 0$ for $\theta \in [0, \theta_1]$ and $\theta \in [\pi - \theta_1, \pi]$. By Condition 1 of Theorem 2, Eq. (2.3) implies that a number r_2 exists such, that $F(r, \pi / 2) > 0$ when $r > r_2$. But in this case a number θ_0 can be found such that $F(r, \theta) > 0$ for $\theta \in (\pi / 2 - \theta_0, \pi / 2 + \theta_0)$ and $r > r_2$.

It remains to confirm that $F(r, \theta) > 0$ for sufficiently large r when $\theta \in (\theta_1, \pi / 2 - \theta_0)$ and $\theta \in (\pi / 2 + \theta_0, \pi - \theta_1)$. By Condition 2 of Theorem 2, a number r_3 can be found for any $\varepsilon > 0$ such, that for all θ belonging to these two intervals when $|r\cos\theta| > r_3$, and consequently for $r > r_3$, the following inequality holds:

$$|f_k(r\cos\theta)| < r^{2m-k} |\cos\theta|^{2m-k} \varepsilon < r^{2m-k} \varepsilon \qquad (k = 1, 2, \ldots, 2m - 1)$$

For these r and θ we have

$$\left|\sum_{k=1}^{2m-1} r^k \sin^k \theta f_k (r\cos\theta)\right| \leqslant r^{2m}\varepsilon_1, \qquad \varepsilon_1 = (2m-1)\varepsilon \tag{2.4}$$

Clearly, in each of these two intervals $\sin\theta > \sin\theta_1$. Therefore, taking the Condition 1 into account, we have

$$\sin^{2m}\theta f_{2m}(r\cos\theta) - \varepsilon_1 > \sin^{2m}\theta_1 \cdot K - \varepsilon_1$$

Setting $\varepsilon_1 < K\sin^{2m}\theta_1$ we obtain

$$\sin^{2m}\theta f_{2m}(r\cos\theta) - \varepsilon_1 > 0 \tag{2.5}$$

with $r > r_3$, $\theta \in (\theta_1, \pi/2 - \theta_0)$ and $\theta \in (\pi/2 + \theta_0, \pi - \theta_1)$.

By Condition 3 a number $M > 0$ exists such, that

$$f_0(r\cos\theta) + M \geqslant 0.$$

for any values of the product $r\cos\theta$, i.e. for any r and θ. Taking into account this inequality as well as (1.5) we can assert, that a number r_4 can be found such that when $r > r_4$,

$$r^{2m}[\sin^{2m}\theta f_{2m}(r\cos\theta) - \varepsilon_1] + [f_0(r\cos\theta) + M] - M > 0 \tag{2.6}$$

From (2.4) and (2.6) follows

$$r^{2m}\sin^{2m}\theta f_{2m}(r\cos\theta) + f_0(r\cos\theta) > r^{2m}\varepsilon_1 \geqslant \left|\sum_{k=1}^{2m-1} r^k \sin^k\theta f_k(r\cos\theta)\right| \tag{2.7}$$

for $r > \max\{r_3, r_4\}$, $\theta \in (\theta_1, \pi/2 - \theta_0)$ and $\theta \in (\pi/2 + \theta_0, \pi - \theta_1)$. By equality (2.3) which defines the function $F(r, \theta)$, the inequality (2.7) is equivalent to the statement that $F(r, \theta) > 0$ for the r and θ shown above. Setting $R_1 = \max\{r_1, r_2, r_3, r_4\}$ we find that $f(r, \theta) > 0$ for $\theta \in [0, \pi]$ and $r > R_1$. Similarly, we can prove that a number R_2 exists such that $F(r, \theta) > 0$ for $r > R_2$ and $\pi \leqslant \theta \leqslant 2\pi$.

Thus if $R = \max\{R_1, R_2\}$, then $F(r, \theta) > 0$ for $r > R$ and $\theta \in [0, 2\pi]$. This means that $f(x, y) > 0$ on any circle with center at the coordinate origin and of radius $r > R$, i.e. $f(x, y) > 0$ at all points of the plane lying outside the circle of radius R. In particular, $f(x, y) > 0$ on any curve belonging to the family $\Phi(x, y) = C$ lying outside this circle. But then , according to (2.2), on these curves we have $d\Phi / dt \leqslant 0$. On the other hand, by Condition 3 $f(0, 0) = f_0(0) < 0$. Consequently $f(x, y) < 0$ in sufficiently small vicinity of the coordinate origin and, according to (2.2), $d\Phi / dt \geqslant 0$ on the curves $\Phi(x, y) = C$ for sufficiently small values of C.

Taking together one of these curves and any other curve of the same family lying outside the circle of radius R we obtain an annular region contained between these curves, and all trajectories of (2.1) are directed into this annulus. Since this region has no singularities, it contains at least one stable limit cycle.

BIBLIOGRAPHY

1. Zhitel'zeif, E. D., On a theorem of existence of a periodic solution to the Liénard equation. PMM Vol. 35, №2, 1971.

Translated by L.K.